A non-linear theory for oscillations in a parallel flow

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Three-dimensional periodic oscillations in the shear flow region between two parallel streams are considered up to that second order of the oscillation amplitude. It is shown that, as an integral part of the oscillation, there is a mean secondary flow in the nature of a longitudinal vortex. Despite the dissimilarity in the profile of the basic flows, several of the principal features of the calculated results can be compared with those observed for the Blasius flow by Schubauer and Klebanoff & Tidstrom at the National Bureau of Standards.

1. Introduction

One of the most basic and challenging problems in fluid mechanics is to obtain an understanding of the various physical mechanisms involved during the transition from laminar to turbulent flow. It could be called the missing link between the two régimes of fluid motion. This problem has been subject to a great deal of theoretical and experimental research, especially over the last decade, and considerable progress has been made to bridge the gap. However, there remains a vast amount of work still to be done before our knowledge of the transition phenomenon is complete. One may anticipate that the final answer will include many simultaneous effects.

In order to study the breakdown of laminar flow it is necessary to follow the growth of a disturbance superposed on the basic flow. If this disturbance is of very small amplitude, the equations can be linearized and one can develop the linear theory of hydrodynamic stability. This theory has been investigated in great detail and a survey of the subject is given in the monograph by Lin (1955). There can be no doubt that the initial trend of a small disturbance will be described adequately by the results of the linearized theory. Indeed there is now ample experimental evidence to support this fact (Schubauer & Skramstead 1948). However, as the oscillation grows the non-linear terms in the equations become important, and must be included in the investigation.

It has long been recognized that the inclusion of the non-linear terms adds two important new features to the problem. First, there is the effect of the Reynolds stresses in producing a redistribution of momentum and so a distortion of the original velocity profile, and secondly, the excitation of higher harmonics of the original oscillation. For finite-amplitude oscillations the modification of the basic flow through the action of the Reynolds stresses can be quite appreciable, and this will in turn modify the rate of transfer of energy from the mean flow to the disturbance, and so the rate of growth of the disturbance. Meksyn &

Stuart (1951) have calculated these modifications for the case of flow between parallel plates, and have given some calculations showing that the production of higher harmonics plays a less important role. Their results show that an increase in the amplitude of the oscillation produces a lower critical Reynolds number. More recently Stuart (1958, 1959) has given a somewhat simpler analysis based on energy methods and has obtained good agreement with experiment for the case of flow between rotating circular cylinders. Both these discussions consider only two-dimensional disturbances. The importance of the critical layer region as the 'weak spot' of instability has been stressed by Lin (1957). He has shown that for disturbances in a parallel flow all the harmonic components of the oscillation simultaneously become important around the critical layer, before the amplitude of the fundamental is large enough to cause any significant distortion of the mean flow, sufficiently large for experimental observation.

The search for a suitable mechanism to describe the onset of turbulence has aroused much interest. Several plausible theories have been proposed and all possess some element of truth, although no single one appears to be the complete answer. Landau's concept (1944) of successive instabilities seems intuitively reasonable, and enables one to picture the appearance of additional modes of oscillation corresponding to a sequence of critical Reynolds numbers. Görtler & Witting (1957) have proposed a theory, in line with Landau's conjecture, based on the curvature of the streamlines causing a periodic vortex structure. There is experimental evidence to support the existence of secondary vortices, although there has not been any definite confirmation of the phase relationships involved.

A horseshoe vortex structure as the fundamental element of transition has been proposed by Theodorson (1952), who also suggested that strictly two-dimensional disturbances are unimportant for causing transition. This latter conjecture is strongly supported by the recent experiments of Schubauer (1957) and Klebanoff & Tidstrom (1958), which we shall discuss briefly below.

The presence of longitudinal vortices during transition has been reported by many experiments. These can be observed using dye and china-clay techniques.

Mention must be made of the relative importance of two- and three-dimensional disturbances. This is a current issue that has attracted much attention. On the basis of linearized theory, Squire's result (1933), namely, that three-dimensional disturbances are equivalent to two-dimensional ones at a lower Reynolds number is applicable, and so to estimate the onset of instability one need only consider two-dimensional disturbances. However, once the flow is above the critical Reynolds number, oblique waves also become unstable. Therefore it must be anticipated that the initially two-dimensional waves will become progressively three-dimensional as the Reynolds number is increased. This point has been emphasized in earlier work by Lin (1957). Indeed, since turbulence is an essentially three-dimensional phenomenon there must be a stage during development when the three-dimensional disturbances tend to dominate. Two-dimensional theory cannot be expected to suffice. This simple observation suggests the necessity of a theoretical investigation of three-dimensional effects.

Recent experiments also point strongly to the desirability of such an investigation. Notable among the vast array of experiments probing the phenomena of transition, is the work of Schaubauer (1957) and Klebanoff & Tidstrom (1958) at the National Bureau of Standards. Most of their work has concerned boundary-layer transition on a flat plate. Perhaps the most startling and significant fact revealed by these experiments is the almost periodic spanwise variations of intensity with peaks and valleys occuping fixed positions and forming streets of high and low intensity. This periodic spanwise variation causes a warping of the velocity profile, the turbulence appearing to originate at these peaks and to spread into the valleys. More recently, further experimental work on this spanwise variation has been done and we shall have occasion to refer to it at a later stage.

This brief introduction points to the multitude of effects observed and predicted during transition. If it serves no other purpose, at least it does pose the question as to whether there is any advantage in a theoretical approach which does not include all the non-linear terms. The complete solution of the non-linear equations should automatically include all of these effects. A consideration of all the non-linear terms has been advocated by von Kármán, and it is in this spirit that we have undertaken the present investigation.

Our task is to examine finite-amplitude disturbances, paying special attention to the three-dimensional oscillations. It is to be stressed that this will be done by setting up a systematic perturbation from the linear theory and that a purely formal mathematical approach is adopted, although much of the motivation for this work stems from recent experimental evidence.

It is difficult to give a detailed explanation of the conclusions before the actual calculations have been made. Therefore at this stage only brief comment will be made on the interpretation of the results. A detailed description will be given later.

The quantity found to be of prime importance is the mean secondary vorticity in the downstream direction. This vorticity has a periodic spanwise variation and produces a redistribution of momentum in planes perpendicular to the direction of flow. It is this momentum exchange that is responsible for an alternate steepening and flattening of the velocity profile, causing a warping or crumbling effect on the basic flow. Explicit formulas are obtained for the rate of growth of the second-order mean motion. Superposed on these secondary vortices there is the vorticity of the primary oscillation itself. This is periodic in the downstream direction, and so the two effects combined should produce alternately partial reinforcement and cancellation over each wavelength.

The results obtained are applicable to a general parallel flow; but for illustrative purposes we have restricted the detailed calculations to the case of a shear profile. It is to be noted that, although our interests are chiefly with the three-dimensional nature of the motion, we do not discount two-dimensional effects. The results found by Meksyn & Stuart (1951) are in fact included in the analysis. It is believed, however, that in most situations the spanwise profile distortion will be the more important mechanism during transition.

2. Mathematical formulation

We now proceed to give the mathematical formulation on which the subsequent calculations will be based. The symbols p, $\mathbf{q}$, $\boldsymbol{\omega}$ and R will be used to denote the fluid pressure, velocity, vorticity and Reynolds number, respectively. As mentioned in § 1, the calculation of the second-order vorticity will play a key role in this development. For this purpose it would suffice to use the vorticity equation, namely,

$$\frac{\partial \boldsymbol{\omega}}{\partial t}+(\mathbf{q}.\nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega}.\nabla)\,\mathbf{q}+\frac{1}{R}\Delta\boldsymbol{\omega}. \tag{2.1}$$

This will be referred to at the end of this section. But in order to calculate the second-order velocities it is convenient to proceed directly from the equations of motion, together with the continuity equation, that is,

$$\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\,\mathbf{q} = -\nabla p+\frac{1}{R}\Delta\mathbf{q}, \tag{2.2}$$

$$\nabla.\mathbf{q} = 0, \tag{2.3}$$

where we have supposed the fluid to be incompressible and all quantities are expressed in dimensionless form.

We may suppose that $\mathbf{q}$, p and $\boldsymbol{\omega}$ are expanded as perturbation series of the form

$$\mathbf{q} = \mathbf{q}^{(0)}+a\mathbf{q}^{(1)}+a^2\mathbf{q}^{(2)}+\ldots, \tag{2.4}$$

$$p = p^{(0)}+ap^{(1)}+a^2p^{(2)}+\ldots, \tag{2.5}$$

$$\boldsymbol{\omega} = \boldsymbol{\omega}^{(0)}+a\boldsymbol{\omega}^{(1)}+a^2\boldsymbol{\omega}^{(2)}+\ldots, \tag{2.6}$$

where $\mathbf{q}^{(0)}$ is the undisturbed basic flow and $\mathbf{q}^{(1)}$ the primary oscillation, etc. The symbol a is used to denote a perturbation amplitude, and $p^{(0)}$ is the pressure distribution associated with the basic flow.

Taking (x, y, z) as rectangular co-ordinates, and a given parallel basic flow $\mathbf{q}^{(0)} = \{u_0^{(0)}(y), 0, 0\}$, we wish to trace the growth of a wave of small amplitude propagating in the x-direction, having a possible z-variation of amplitude, which will for the moment remain unspecified.

Successive perturbations are determined by sets of equations of the type

$$\frac{\partial \mathbf{q}^{(n)}}{\partial t}+\sum_{r=0}^{n}(\mathbf{q}^{(n-r)}.\nabla)\,\mathbf{q}^{(r)} = -\nabla p^{(n)}+\frac{1}{R}\Delta\mathbf{q}^{(n)}, \tag{2.7}$$

$$\nabla.\mathbf{q}^{(n)} = 0 \quad (n = 1, 2, \ldots). \tag{2.8}$$

We now proceed to derive the equations relevant for a determination of $\mathbf{q}$ and $\boldsymbol{\omega}$ up to terms $O(a^2)$, and in particular to find the second-order mean motion induced by the primary oscillation.

Let

$$\mathbf{q} = (u, v, w), \tag{2.9}$$

$$\mathbf{q}^{(n)} = (u^{(n)}, v^{(n)}, w^{(n)}), \tag{2.10}$$

and so

$$\boldsymbol{\omega} = (\xi, \eta, \zeta) = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right), \tag{2.11}$$

$$\boldsymbol{\omega}^{(n)} = (\xi^{(n)}, \eta^{(n)}, \zeta^{(n)}) = \left(\frac{\partial w^{(n)}}{\partial y}-\frac{\partial v^{(n)}}{\partial z}, \frac{\partial u^{(n)}}{\partial z}-\frac{\partial w^{(n)}}{\partial x}, \frac{\partial v^{(n)}}{\partial x}-\frac{\partial u^{(n)}}{\partial y}\right). \tag{2.12}$$

If α is the wave-number of the primary oscillation associated with the x-direction, we take,

$$\left.\begin{aligned}
u^{(1)} &= u_1^{(1)}(y,z,t)\,e^{i\alpha x}+u_1^{(1)*}(y,z,t)\,e^{-i\alpha x},\\
v^{(1)} &= v_1^{(1)}(y,z,t)\,e^{i\alpha x}+v_1^{(1)*}(y,z,t)\,e^{-i\alpha x},\\
w^{(1)} &= w_1^{(1)}(y,z,t)\,e^{i\alpha x}+w_1^{(1)*}(y,z,t)\,e^{-i\alpha x},\\
p^{(1)} &= p_1^{(1)}(y,z,t)\,e^{i\alpha x}+p_1^{(1)*}(y,z,t)\,e^{-i\alpha x},\\
\xi^{(1)} &= \xi_1^{(1)}(y,z,t)\,e^{i\alpha x}+\xi_1^{(1)*}(y,z,t)\,e^{-i\alpha x},\\
\eta^{(1)} &= \eta_1^{(1)}(y,z,t)\,e^{i\alpha x}+\eta_1^{(1)*}(y,z,t)\,e^{-i\alpha x},\\
\zeta^{(1)} &= \zeta_1^{(1)}(y,z,t)\,e^{i\alpha x}+\zeta_1^{(1)*}(y,z,t)\,e^{-i\alpha x},
\end{aligned}\right\} \tag{2.13}$$

where

$$\left.\begin{aligned}
\xi_1^{(1)} &= \frac{\partial w_1^{(1)}}{\partial y}-\frac{\partial v_1^{(1)}}{\partial z},\\
\eta_1^{(1)} &= \frac{\partial u_1^{(1)}}{\partial z}-i\alpha w_1^{(1)},\\
\zeta_1^{(1)} &= i\alpha v_1^{(1)}-\frac{\partial u_1^{(1)}}{\partial y},
\end{aligned}\right\} \tag{2.14}$$

and where an asterisk is used to denote a complex conjugate.

The equations governing the first-order motion are

$$\left.\begin{aligned}
\frac{\partial u_1^{(1)}}{\partial t}+i\alpha u_0^{(0)}u_1^{(1)}+v_1^{(1)}\frac{\partial u_0^{(0)}}{\partial y} &= -i\alpha p_1^{(1)}+\frac{1}{R}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\alpha^2\right)u_1^{(1)},\\
\frac{\partial v_1^{(1)}}{\partial t}+i\alpha u_0^{(0)}v_1^{(1)} &= -\frac{\partial p_1^{(1)}}{\partial y}+\frac{1}{R}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\alpha^2\right)v_1^{(1)},\\
\frac{\partial w_1^{(1)}}{\partial t}+i\alpha u_0^{(0)}w_1^{(1)} &= -\frac{\partial p_1^{(1)}}{\partial z}+\frac{1}{R}\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\alpha^2\right)w_1^{(1)},\\
i\alpha u_1^{(1)}+\frac{\partial v_1^{(1)}}{\partial y}+\frac{\partial w_1^{(1)}}{\partial z} &= 0.
\end{aligned}\right\} \tag{2.15}$$

The secondary flow $\mathbf{q}^{(2)}$, $p^{(2)}$, $\boldsymbol{\omega}^{(2)}$, can be written as the sum of a mean motion and a second-harmonic oscillation. Thus,

$$\left.\begin{aligned}
u^{(2)} &= u_0^{(2)}(y,z,t)+u_2^{(2)}(y,z,t)\,e^{2i\alpha x}+u_2^{(2)*}(y,z,t)\,e^{-2i\alpha x},\\
v^{(2)} &= v_0^{(2)}(y,z,t)+v_2^{(2)}(y,z,t)\,e^{2i\alpha x}+v_2^{(2)*}(y,z,t)\,e^{-2i\alpha x},\\
w^{(2)} &= w_0^{(2)}(y,z,t)+w_2^{(2)}(y,z,t)\,e^{2i\alpha x}+w_2^{(2)*}(y,z,t)\,e^{-2i\alpha x},\\
p^{(2)} &= p_0^{(2)}(y,z,t)+p_2^{(2)}(y,z,t)\,e^{2i\alpha x}+p_2^{(2)*}(y,z,t)\,e^{-2i\alpha x},\\
\xi^{(2)} &= \xi_0^{(2)}(y,z,t)+\xi_2^{(2)}(y,z,t)\,e^{2i\alpha x}+\xi_2^{(2)*}(y,z,t)\,e^{-2i\alpha x},\\
\eta^{(2)} &= \eta_0^{(2)}(y,z,t)+\eta_2^{(2)}(y,z,t)\,e^{2i\alpha x}+\eta_2^{(2)*}(y,z,t)\,e^{-2i\alpha x},\\
\zeta^{(2)} &= \zeta_0^{(2)}(y,z,t)+\zeta_2^{(2)}(y,z,t)\,e^{2i\alpha x}+\zeta_2^{(2)*}(y,z,t)\,e^{-2i\alpha x},
\end{aligned}\right\} \tag{2.16}$$

where

$$\begin{aligned}
\xi_0^{(2)} &= \frac{\partial w_0^{(2)}}{\partial y}-\frac{\partial v_0^{(2)}}{\partial z},\\
\eta_0^{(2)} &= \frac{\partial u_0^{(2)}}{\partial z},\\
\zeta_0^{(2)} &= -\frac{\partial u_0^{(2)}}{\partial y},
\end{aligned}$$

and
$$\xi_2^{(2)} = \frac{\partial w_2^{(2)}}{\partial y} - \frac{\partial v_2^{(2)}}{\partial z},$$
$$\eta_2^{(2)} = \frac{\partial u_2^{(2)}}{\partial z} - 2i\alpha w_2^{(2)},$$
$$\zeta_2^{(2)} = 2i\alpha v_2^{(2)} - \frac{\partial u_2^{(2)}}{\partial y}.$$

The notation used is such that superscripts refer to the order of the motion and subscripts give the harmonic.

The equations to determine the mean secondary flow are

$$\left.\begin{aligned}
&\frac{\partial u_0^{(2)}}{\partial t} + v_0^{(2)}\frac{du_0^{(0)}}{dy} + \frac{\partial}{\partial y}(u_1^{(1)}v_1^{(1)*} + u_1^{(1)*}v_1^{(1)}) + \frac{\partial}{\partial z}(u_1^{(1)}w_1^{(1)*} + u_1^{(1)*}w_1^{(1)}) \\
&\qquad = \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u_0^{(2)}, \\
&\frac{\partial v_0^{(2)}}{\partial t} + 2\frac{\partial}{\partial y}(v_1^{(1)}v_1^{(1)*}) + \frac{\partial}{\partial z}(v_1^{(1)}w_1^{(1)*} + v_1^{(1)*}w_1^{(1)}) = -\frac{\partial p_0^{(2)}}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)v_0^{(2)}, \\
&\frac{\partial w_0^{(2)}}{\partial t} + \frac{\partial}{\partial y}(v_1^{(1)}w_1^{(1)*} + v_1^{(1)*}w_1^{(1)}) + 2\frac{\partial}{\partial z}(w_1^{(1)}w_1^{(1)*}) = -\frac{\partial p_0^{(2)}}{\partial z} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)w_0^{(2)}, \\
&\frac{\partial v_0^{(2)}}{\partial y} + \frac{\partial w_0^{(2)}}{\partial z} = 0.
\end{aligned}\right\} \quad (2.17)$$

The second-order oscillation satisfies

$$\left.\begin{aligned}
&\frac{\partial u_2^{(2)}}{\partial t} + v_2^{(2)}\frac{du_0^{(0)}}{dy} + 2i\alpha u_0^{(0)}u_2^{(2)} + i\alpha u_1^{(1)2} + v_1^{(1)}\frac{\partial u_1^{(1)}}{\partial y} + w_1^{(1)}\frac{\partial u_1^{(1)}}{\partial z} \\
&\qquad = -2i\alpha p_2^{(2)} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 4\alpha^2\right)u_2^{(2)}, \\
&\frac{\partial v_2^{(2)}}{\partial t} + 2i\alpha u_0^{(0)}v_2^{(2)} + i\alpha u_1^{(1)}v_1^{(1)} + v_1^{(1)}\frac{\partial v_1^{(1)}}{\partial y} + w_1^{(1)}\frac{\partial v_1^{(1)}}{\partial z} \\
&\qquad = -\frac{\partial p_2^{(2)}}{\partial y} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 4\alpha^2\right)v_2^{(2)}, \\
&\frac{\partial w_2^{(2)}}{\partial t} + 2i\alpha u_0^{(0)}w_2^{(2)} + i\alpha u_1^{(1)}w_1^{(1)} + v_1^{(1)}\frac{\partial w_1^{(1)}}{\partial y} + w_1^{(1)}\frac{\partial w_1^{(1)}}{\partial z} \\
&\qquad = -\frac{\partial p_2^{(2)}}{\partial z} + \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - 4\alpha^2\right)w_2^{(2)}, \\
&2i\alpha u_2^{(2)} + \frac{\partial v_2^{(2)}}{\partial y} + \frac{\partial w_2^{(2)}}{\partial z} = 0.
\end{aligned}\right\} \quad (2.18)$$

From the set (2.17), or from (2.1), it is readily seen that the component $\xi_0^{(2)}$ of the mean second-order vorticity satisfies the equation

$$\frac{\partial \xi_0^{(2)}}{\partial t} + \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\right)(v_1^{(1)}w_1^{(1)*} + v_1^{(1)*}w_1^{(1)}) + 2\frac{\partial^2}{\partial y\,\partial z}(w_1^{(1)}w_1^{(1)*} - v_1^{(1)}v_1^{(1)*}) = \frac{1}{R}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\xi_0^{(2)}. \quad (2.19)$$

The momentum transfer associated with this longitudinal vorticity will be of prime importance in later discussions.

3. Distortion effects at high Reynolds numbers

For flows at high Reynolds numbers the velocity field will conform to the inviscid equations as a first approximation, except within the critical layer region, where viscous corrections would be needed. A discussion of the subtleties involved in this limiting process can be found in Chapter 8 of Lin's book (1955). Indeed the inviscid limit is justifiable for the case of amplified disturbances and for neutral disturbances as a limiting case. If the complex wave velocity is $c = c_r + ic_i$, then c_r represents the wave speed, $c_i > 0$ implies amplified, $c_i = 0$ neutral, and $c_i < 0$ damped disturbances. It is convenient to introduce the amplitude functions for the primary oscillation; these are denoted by a circumflex. Thus,

$$u_1^{(1)}(y,z,t) = \hat{u}_1^{(1)}(y,z)\,e^{-i\alpha ct}, \tag{3.1}$$

and similarly for $v_1^{(1)}, w_1^{(1)}$, and $p_1^{(1)}$.

In studying the growth or decay of an oscillation we are interested in the case when c_i is close to zero. For the case of large Reynolds numbers we can find quite simple explicit formulas for the rates of growth of the second-order mean velocity modifications. To this purpose it is convenient to rewrite equations (2.17). Dropping the viscous terms, we have

$$\frac{\partial v_0^{(2)}}{\partial y} + \frac{\partial w_0^{(2)}}{\partial z} = 0, \tag{3.2}$$

$$\frac{\partial w_0^{(2)}}{\partial y} - \frac{\partial v_0^{(2)}}{\partial z} = \xi_0^{(2)}, \tag{3.3}$$

$$\frac{\partial u_0^{(2)}}{\partial t} + v_0^{(2)}\frac{du_0^{(0)}}{dy} - \hat{r}_0^{(2)}(y,z)\,e^{2\alpha c_i t} = 0, \tag{3.4}$$

$$\frac{\partial \xi_0^{(2)}}{\partial t} = \hat{\xi}_0^{(2)}(y,z)\,e^{2\alpha c_i t}, \tag{3.5}$$

where
$$\hat{r}_0^{(2)}(y,z) = -\frac{\partial}{\partial y}(\hat{u}_1^{(1)}\hat{v}_1^{(1)*} + \hat{u}_1^{(1)*}\hat{v}_1^{(1)}) - \frac{\partial}{\partial z}(\hat{u}_1^{(1)}\hat{w}_1^{(1)*} + \hat{u}_1^{(1)*}\hat{w}_1^{(1)}), \tag{3.6}$$

and
$$\hat{\xi}_0^{(2)}(y,z) = \left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial y^2}\right)(\hat{v}_1^{(1)}\hat{w}_1^{(1)*} + \hat{v}_1^{(1)*}\hat{w}_1^{(1)}) + 2\frac{\partial^2}{\partial y\,\partial z}(\hat{v}_1^{(1)}\hat{v}_1^{(1)*} - \hat{w}_1^{(1)}\hat{w}_1^{(1)*}). \tag{3.7}$$

Integrating (3.5), we have

$$\xi_0^{(2)} = \hat{\xi}_0^{(2)}(y,z)\left(\frac{e^{2\alpha c_i t} - 1}{2\alpha c_i}\right) + \hat{C}(y,z), \tag{3.8}$$

where the form of the solution is so written that the limiting case $c_i \to 0$ can readily be discussed. Clearly the arbitrary function of integration gives the initial value of $\xi_0^{(2)}$, which we may take as zero for the purpose at hand. The interpretation of the solution for $t = 0$ is to be taken more in the formal rather than the physical sense. The solution is supposed to be valid for a range of t, $t_1 < t < t_2$.†

† When the manuscript was shown to Dr J. T. Stuart and Mr J. Watson, they pointed out that the method they have developed for two-dimensional disturbances can be adopted here to modify the solution so as to make it valid for $t \to -\infty$.

In using this result for $\xi_0^{(2)}$, the time dependence of the corresponding velocity components can easily be found. The results are

$$u_0^{(2)} = \hat{s}_0^{(2)}(y,z)\,\frac{(e^{2\alpha c_i t}-1-2\alpha c_i t)}{(2\alpha c_i)^2}+\hat{r}_0^{(2)}(y,z)\left(\frac{e^{2c\alpha_i t}-1}{2\alpha c_i}\right), \tag{3.9}$$

$$v_0^{(2)} = \hat{v}_0^{(2)}(y,z)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right), \tag{3.10}$$

$$w_0^{(2)} = \hat{w}_0^{(2)}(y,z)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right), \tag{3.11}$$

$$\xi_0^{(2)} = \hat{\xi}_0^{(2)}(y,z)\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right), \tag{3.12}$$

where

$$\frac{\partial \hat{v}_0^{(2)}}{\partial y}+\frac{\partial \hat{w}_0^{(2)}}{\partial z} = 0, \tag{3.13}$$

$$\frac{\partial \hat{w}_0^{(2)}}{\partial y}-\frac{\partial \hat{v}_0^{(2)}}{\partial z} = \hat{\xi}_0^{(2)}(y,z), \tag{3.14}$$

$$\hat{s}_0^{(2)}(y,z) = -\hat{v}_0^{(2)}(y,z)\,\frac{du_0^{(0)}}{dy}, \tag{3.15}$$

and $\hat{r}_0^{(2)}(y,z)$, $\hat{\xi}_0^{(2)}(y,z)$ are given by (3.6) and (3.7).

A knowledge of the primary oscillation is sufficient to determine $\hat{r}_0^{(2)}(y,z)$ and $\hat{\xi}_0^{(2)}(y,z)$, thence the solution of a Poisson equation gives the second-order mean velocities. The equations also show that the second-order mean velocities grow exponentially in time, or in the case of neutral oscillations as powers of t; for

$$\lim_{c_i\to 0}\left(\frac{e^{2\alpha c_i t}-1}{2\alpha c_i}\right) = t,$$

$$\lim_{c_i\to 0}\left\{\frac{e^{2\alpha c_i t}-1-2\alpha c_i t}{(2\alpha c_i)^2}\right\} = \tfrac{1}{2}t^2.$$

Therefore even for neutral disturbances the secondary flow will build up and eventually dominate.

One of the most interesting features of this analysis is that a three-dimensional primary oscillation induces a mean second-order vorticity into the flow, having a component $\xi_0^{(2)}$ in the downstream direction. It is this mechanism which produces a spanwise momentum exchange and causes a warping of the original velocity profile.

Thus far the analysis has been kept quite general and the equations would apply to a general type of parallel flow. To be more definite we now prescribe the primary oscillation. The initial oscillation must be expected to be two-dimensional. However, as mentioned earlier in § 1, three-dimensional waves must appear at some stage during the development. We assume this to occur before the oscillation leaves the linear range. There is a strong experimental support for this conjecture. Schubauer and Klebanoff in their experiments report that transition never occurs without first being preceded by a strong warping of the wave.

A more general infinitesimal oscillation can be compounded of a purely two-dimensional component together with a standing wave in the spanwise direction.

Of course, in the practical situation this latter component must be present to some degree, for undoubtedly there will always be some spanwise irregularity in the amplitude of the oscillation. The question then arises as to which component will be dominant as transition is approached. This will depend on which component has the higher amplification rate and so would require the details of the stability for the flow under consideration.

We now revert to a less cumbersome notation and consider in detail the case of a sinusoidal spanwise variation of amplitude (wave-number β) associated with the three-dimensional wave. We write

$$\left.\begin{aligned} u_0^{(0)}(y) &= u_0(y), \\ u_1^{(1)}(y,z,t) &= \hat{u}_1^{(1)}(y,z)\,e^{-i\alpha ct} = [\lambda\hat{u}_1(y)\cos\beta z+\mu\hat{U}_1(y)]\,e^{-i\alpha ct}, \\ v_1^{(1)}(y,z,t) &= \hat{v}_1^{(1)}(y,z)\,e^{-i\alpha ct} = [\lambda\hat{v}_1(y)\cos\beta z+\mu\hat{V}_1(y)]\,e^{-i\alpha ct}, \\ w_1^{(1)}(y,z,t) &= \hat{w}_1^{(1)}(y,z)\,e^{-i\alpha ct} = [\lambda\hat{w}_1(y)\sin\beta z]\,e^{-i\alpha ct}, \\ p_1^{(1)}(y,z,t) &= \hat{p}_1^{(1)}(y,z)\,e^{-i\alpha ct} = [\lambda\hat{p}_1(y)\cos\beta z+\mu\hat{P}_1(y)]\,e^{-i\alpha ct}, \\ \xi_1^{(1)}(y,z,t) &= \hat{\xi}_1^{(1)}(y,z)\,e^{-i\alpha ct} = [\lambda\hat{\xi}_1(y)\sin\beta z]\,e^{-i\alpha ct}, \end{aligned}\right\} \tag{3.16}$$

where the ratio μ/λ is a measure of the relative importance of two- and three-dimensional oscillations, and will be taken to be real. There is then no loss of generality in taking this ratio to be positive. The reality of μ/λ corresponds to taking the two- and three-dimensional oscillations to be in phase. This corresponds to the situation in the experiments of Schubauer and Klebanoff. Also it appears to be the most likely physical situation if one regards the three-dimensionality developing as a perturbation from the two-dimensional waves.†

The two sets of equations governing the primary oscillations then become

$$\left.\begin{aligned} i\alpha\hat{u}_1+\frac{d\hat{v}_1}{dy}+\beta\hat{w}_1 &= 0, \\ i\alpha(u_0-c)\,\hat{u}_1+\frac{du_0}{dy}\hat{v}_1 &= -i\alpha\hat{p}_1, \\ i\alpha(u_0-c)\,\hat{v}_1 &= -\frac{d\hat{p}_1}{dy}, \\ i\alpha(u_0-c)\,\hat{w}_1 &= +\beta\hat{p}_1, \end{aligned}\right\} \tag{3.17}$$

and

$$\left.\begin{aligned} i\alpha\hat{U}_1+\frac{d\hat{V}_1}{dy} &= 0, \\ i\alpha(u_0-c)\,\hat{U}_1+\frac{du_0}{dy}\hat{V}_1 &= -i\alpha\hat{P}_1, \\ i\alpha(u_0-c)\,\hat{V}_1 &= -\frac{d\hat{P}_1}{dy}. \end{aligned}\right\} \tag{3.18}$$

Also, the function $\hat{\xi}_1(y)$ is determined by

$$\hat{\xi}_1 = \frac{d\hat{w}_1}{dy}+\beta\hat{v}_1. \tag{3.19}$$

† If μ/λ is complex the subsequent analysis suffers some slight modifications. These will not be considered in this paper.

Corresponding to this primary oscillation, the mean secondary flow has components which we write as

$$\left.\begin{aligned}\hat{v}_0^{(2)}(y,z) &= \lambda^2\hat{v}_a(y)\cos 2\beta z+\lambda\mu\hat{v}_b(y)\cos\beta z+\lambda^2\hat{v}_c(y)+\mu^2\hat{v}_d(y),\\ \hat{w}_0^{(2)}(y,z) &= \lambda^2\hat{w}_a(y)\sin 2\beta z+\lambda\mu\hat{w}_b(y)\sin\beta z+\lambda^2\hat{w}_c(y)+\mu^2\hat{w}_d(y),\\ \hat{\xi}_0^{(2)}(y,z) &= \lambda^2\hat{\xi}_a(y)\sin 2\beta z+\lambda\mu\hat{\xi}_b(y)\sin\beta z+\lambda^2\hat{\xi}_c(y)+\mu^2\hat{\xi}_d(y),\\ \hat{r}_0^{(2)}(y,z) &= \lambda^2\hat{r}_a(y)\cos 2\beta z+\lambda\mu\hat{r}_b(y)\cos\beta z+\lambda^2\hat{r}_c(y)+\mu^2\hat{r}_d(y),\\ \hat{s}_0^{(2)}(y,z) &= \lambda^2\hat{s}_a(y)\cos 2\beta z+\lambda\mu\hat{s}_b(y)\cos\beta z+\lambda^2\hat{s}_c(y)+\mu^2\hat{s}_d(y),\\ \hat{p}_0^{(2)}(y,z) &= \lambda^2\hat{p}_a(y)\cos 2\beta z+\lambda\mu\hat{p}_b(y)\cos\beta z+\lambda^2\hat{p}_c(y)+\mu^2\hat{p}_d(y).\end{aligned}\right\}\quad(3.20)$$

The subscript a refers to that part of the mean secondary flow induced by the purely three-dimensional part of the oscillation, and the subscript b to the part due to the interaction of the two components of the primary oscillation. Both these secondary a and b flows are periodic in z and will produce three-dimensional distortions of the original flow profile. Quantities with subscripts c and d are merely two-dimensional distortions due to the three- and two-dimensional components of the fluctuation respectively. If $\lambda = 0$ the mean secondary flow would reduce to the d flow, corresponding to the case of a purely two-dimensional oscillation which has been considered earlier by Meksyn & Stuart (1951).

On referring to the general equations for the mean secondary flow derived at the beginning of this section we can now obtain, using the assumed primary oscillation, sets of equations to determine the a, b, c, and d flows.

The a flow is determined by

$$\left.\begin{aligned}\hat{s}_a &= -\hat{v}_a\frac{du_0}{dy},\\ \hat{r}_a &= -\frac{1}{2}\frac{d}{dy}(\hat{u}_1\hat{v}_1^*+\hat{u}_1^*\hat{v}_1)-\beta(\hat{u}_1\hat{w}_1^*+\hat{u}_1^*\hat{w}_1),\\ \hat{\xi}_a &= -\left(\frac{1}{2}\frac{d^2}{dy^2}+2\beta^2\right)(\hat{v}_1\hat{w}_1^*+\hat{v}_1^*\hat{w}_1)-2\beta\frac{d}{dy}(\hat{v}_1\hat{v}_1^*+\hat{w}_1\hat{w}_1^*),\\ &\frac{d\hat{v}_a}{dy}+2\beta\hat{w}_a = 0,\\ &\frac{d\hat{w}_a}{dy}+2\beta\hat{v}_a = \hat{\xi}_a.\end{aligned}\right\}\quad(3.21)$$

The b flow is determined by

$$\left.\begin{aligned}\hat{s}_b &= -\hat{v}_b\frac{du_0}{dy},\\ \hat{r}_b &= -\frac{d}{dy}(\hat{u}_1\hat{V}_1^*+\hat{u}_1^*\hat{V}_1+\hat{U}_1\hat{v}_1^*+\hat{U}_1^*\hat{v}_1)-\beta(\hat{U}_1\hat{w}_1^*+\hat{U}_1^*\hat{w}_1),\\ \hat{\xi}_b &= -\left(\frac{d^2}{dy^2}+\beta^2\right)(\hat{V}_1\hat{w}_1^*+\hat{V}_1^*\hat{w}_1)-2\beta\frac{d}{dy}(\hat{V}_1\hat{v}_1^*+\hat{V}_1^*\hat{v}_1),\\ &\frac{d\hat{v}_b}{dy}+\beta\hat{w}_b = 0,\\ &\frac{d\hat{w}_b}{dy}+\beta\hat{v}_b = \hat{\xi}_b.\end{aligned}\right\}\quad(3.22)$$

The c and d flows are independent of z and so $\hat{\xi}_c = \hat{\xi}_d = 0$. Also

$$\frac{d\hat{v}_c}{dy} = \frac{d\hat{v}_d}{dy} = \frac{d\hat{w}_c}{dy} = \frac{d\hat{w}_d}{dy} = 0.$$

Hence, for the case of a fluid in the presence of a solid boundary or of a fluid extending to infinity, these four velocity components must be zero.

Therefore the c flow is given by

$$\left.\begin{aligned} \hat{s}_c &= 0, \\ \hat{r}_c &= -\frac{1}{2}\frac{d}{dy}(\hat{u}_1\hat{v}_1^* + \hat{u}_1^*\hat{v}_1), \\ \hat{\xi}_c &= 0, \\ \hat{v}_c &= 0, \\ \hat{w}_c &= 0. \end{aligned}\right\} \tag{3.23}$$

Similarly the d flow is given by

$$\left.\begin{aligned} \hat{s}_d &= 0, \\ \hat{r}_d &= -\frac{1}{2}\frac{d}{dy}(U_1V_1^* + U_1^*V_1), \\ \hat{\xi}_d &= 0, \\ \hat{v}_d &= 0, \\ \hat{w}_d &= 0. \end{aligned}\right\} \tag{3.24}$$

Thus these two-dimensional distortions contribute to $u_0^{(2)}$ through the gradient of the Reynolds stress, but do not contribute to $v_0^{(2)}$ or $w_0^{(2)}$.

For the sake of completeness we list the equations for the analogous contributions (e, f, g, h) to the oscillatory secondary flow. We write

$$\left.\begin{aligned} u_2^{(2)}(y,z,t) &= \hat{u}_2^{(2)}(y,z)\,e^{-2i\alpha ct} \\ &= [\lambda^2\hat{u}_e(y)\cos 2\beta z + \lambda\mu\hat{u}_f(y)\cos\beta z + \lambda^2\hat{u}_g(y) + \mu^2\hat{u}_h(y)]\,e^{-2i\alpha ct}, \\ v_2^{(2)}(y,z,t) &= \hat{v}_2^{(2)}(y,z)\,e^{-2i\alpha ct} \\ &= [\lambda^2\hat{v}_e(y)\cos 2\beta z + \lambda\mu\hat{v}_f(y)\cos\beta z + \lambda^2\hat{v}_g(y) + \mu^2\hat{v}_h(y)]\,e^{-2i\alpha ct}, \\ w_2^{(2)}(y,z,t) &= \hat{w}_2^{(2)}(y,z)\,e^{-2i\alpha ct} \\ &= [\lambda^2\hat{w}_e(y)\sin 2\beta z + \lambda\mu\hat{w}_f(y)\sin\beta z + \lambda^2\hat{w}_g(y) + \mu^2\hat{w}_h(y)]\,e^{-2i\alpha ct}, \\ p_2^{(2)}(y,z,t) &= \hat{p}_2^{(2)}(y,z)\,e^{-2i\alpha ct} \\ &= [\lambda^2\hat{p}_e(y)\cos 2\beta z + \lambda\mu\hat{p}_f(y)\cos\beta z + \lambda^2\hat{p}_g(y) + \mu^2\hat{p}_h(y)]\,e^{-2i\alpha ct}. \end{aligned}\right\} \tag{3.25}$$

The e flow is then determined by the equations

$$\left.\begin{aligned} 2i\alpha(u_0-c)\,\hat{u}_e + \frac{du_0}{dy}\hat{v}_e + \frac{1}{2}\left(\hat{v}_1\frac{d\hat{u}_1}{dy} - \hat{u}_1\frac{d\hat{v}_1}{dy}\right) &= -2i\alpha\hat{p}_e, \\ 2i\alpha(u_0-c)\,\hat{v}_e &= -\frac{d\hat{p}_e}{dy}, \\ 2i\alpha(u_0-c)\,\hat{w}_e + \frac{1}{2}\left(\hat{v}_1\frac{d\hat{w}_1}{dy} - \hat{w}_1\frac{d\hat{v}_1}{dy}\right) &= 2\beta\hat{p}_e, \\ 2i\alpha\hat{u}_e + \frac{d\hat{v}_e}{dy} + 2\beta\hat{w}_e &= 0, \\ \hat{\xi}_e &= \frac{d\hat{w}_e}{dy} + 2\beta\hat{v}_e. \end{aligned}\right\} \tag{3.26}$$

For the f flow

$$\left.\begin{aligned}2i\alpha(u_0-c)\,\hat{u}_f+\frac{du_0}{dy}\hat{v}_f+2i\alpha\hat{u}_1\hat{U}_1+\hat{v}_1\frac{d\hat{U}_1}{dy}+\hat{V}_1\frac{d\hat{u}_1}{dy} &= -2i\alpha\hat{p}_f,\\ 2i\alpha(u_0-c)\,\hat{v}_f-\beta\hat{w}_1\hat{V}_1 &= -\frac{d\hat{p}_f}{dy},\\ 2i\alpha(u_0-c)\,\hat{w}_f+\hat{V}_1\frac{d\hat{w}_1}{dy}-\hat{w}_1\frac{d\hat{V}_1}{dy} &= \beta\hat{p}_f,\\ 2i\alpha\hat{u}_f+\frac{d\hat{v}_f}{dy}+\beta\hat{w}_f &= 0,\\ \xi_f &= \frac{d\hat{w}_f}{dy}+\beta\hat{v}_f.\end{aligned}\right\} \tag{3.27}$$

For the g flow

$$\left.\begin{aligned}2i\alpha(u_0-c)\,\hat{u}_g+\frac{du_0}{dy}\hat{v}_g+\frac{1}{2}\left(\hat{v}_1\frac{d\hat{u}_1}{dy}-\hat{u}_1\frac{d\hat{v}_1}{dy}\right)-\beta\hat{u}_1\hat{w}_1 &= -2i\alpha\hat{p}_g,\\ 2i\alpha(u_0-c)\,\hat{v}_g-\beta\hat{v}_1\hat{w}_1 &= -\frac{d\hat{p}_g}{dy},\\ \hat{w}_g &= 0,\\ 2i\alpha\hat{u}_g+\frac{d\hat{v}_g}{dy} &= 0,\\ \xi_g &= 0.\end{aligned}\right\} \tag{3.28}$$

Finally, the h flow is given by

$$\left.\begin{aligned}2i\alpha(u_0-c)\,\hat{u}_h+\frac{du_0}{dy}\hat{v}_h+\hat{V}_1\frac{d\hat{U}_1}{dy}-\hat{U}_1\frac{d\hat{V}_1}{dy} &= -2i\alpha\hat{p}_h,\\ 2i\alpha(u_0-c)\,\hat{v}_h &= -\frac{d\hat{p}_h}{dy},\\ \hat{w}_h &= 0,\\ 2i\alpha\hat{u}_h+\frac{d\hat{v}_h}{dy} &= 0,\\ \xi_h &= 0.\end{aligned}\right\} \tag{3.29}$$

We shall now perform the calculations for a particular case. Such a calculation will shed considerable light on the mechanism involved.

4. Application to the case of a shear flow between parallel streams

In this section detailed calculations are given for the case of the shear profile $u_0(y) = \tanh y$ (cf. Esch 1957, and Drazin 1958). Such a shear flow can be readily approximated experimentally by mixing two parallel streams, and so the theoretical predictions should be capable of direct confirmation. At the time of performing these calculations no experimental results for a shear flow, with special attention to spanwise variations, appear to be available. Such experiments would be highly desirable.

From the set of equations (3.17), governing the standing wave component of the primary oscillation, it can be shown by a simple elimination that $\hat{v}_1$ satisfies the equation

$$(u_0-c)\left(\frac{d^2}{dy^2}-\alpha^2-\beta^2\right)\hat{v}_1-\frac{d^2u_0}{dy^2}\hat{v}_1 = 0. \tag{4.1}$$

For $u_0(y) = \tanh y$, we have a point of inflexion at $y = 0$, and Curle (1956) has shown there is a neutral oscillation given by

$$\hat{v}_1 = \operatorname{sech} y; \quad \alpha^2+\beta^2 = 1; \quad c = 0. \tag{4.2}$$

In fact one can set up a regular perturbation in c about this solution using the method of variation of constants. As our interest centres on the case of c_i small, we shall be content with the first term of this perturbation.

A complete solution of the linear system (3.17) corresponding to this neutral oscillation is

$$\left.\begin{aligned}\hat{u}_{1i} &= \frac{i}{\alpha}[\beta^2 \operatorname{cosech} y - \operatorname{sech} y \tanh y],\\ \hat{v}_{1i} &= \operatorname{sech} y,\\ \hat{w}_{1i} &= \beta \operatorname{cosech} y,\\ \hat{p}_{1i} &= i\alpha \operatorname{sech} y.\end{aligned}\right\} \tag{4.3}$$

For the two-dimensional component of the primary oscillation we take the known neutral solution of the set of equations† (3.18), namely,

$$\left.\begin{aligned}\hat{U}_{1i} &= -i \operatorname{sech} y \tanh y,\\ \hat{V}_{1i} &= \operatorname{sech} y,\\ \hat{P}_{1i} &= i \operatorname{sech} y.\end{aligned}\right\} \tag{4.4}$$

These will be reliable solutions for fairly slow spanwise variations of amplitude and for c_i small. Again the solutions could be improved by perturbation techniques; but for the range of interest the above approximations will be taken as adequate. As was mentioned in § 3, in practice one part of the second-order mean flow (the b flow) may be of a quasi-steady nature, corresponding to slightly different downstream wave-numbers for the two- and three-dimensional parts of the primary oscillations.

In the solutions (4.3) and (4.4), the subscript i has been added to emphasize the fact that these solutions are purely inviscid solutions, and therefore will be reliable solutions for the flow of a real fluid at high Reynolds number, provided we are not close to the critical layer ($y = 0$). It is important to note that the amplitude functions $\hat{u}_{1i}$ and $\hat{w}_{1i}$ exhibit singularities at $y = 0$, which would be absent in any real fluid. That is, when viscosity is included these poles would not be present.

On the basis of a purely inviscid analysis, we obtain, from (3.21) to (3.24),

$$\left.\begin{aligned}\hat{\xi}_{ai} &= -\frac{16\beta}{\sinh^3 2y}[(1-\beta^2)\cosh 2y-\beta^2],\\ \hat{r}_{ai} &= 0,\end{aligned}\right\} \tag{4.5}$$

$$\left.\begin{aligned}\hat{\xi}_{bi} &= -\frac{2\beta}{\sinh^3 2y}[16\cosh 2y+\beta^2\cosh 4y-\beta^2],\\ \hat{r}_{bi} &= 0,\end{aligned}\right\} \tag{4.6}$$

$$\left.\begin{aligned}\hat{\xi}_{ci} &= \hat{r}_{ci} = 0,\\ \hat{\xi}_{di} &= \hat{r}_{di} = 0.\end{aligned}\right\} \tag{4.7}$$

† Strictly, this solution would have to be modified due to the difference in the value of α; but the logarithmic singularity is believed to be unimportant compared to the higher-order singularities present in the inviscid oscillation.

It is seen that the functions ξ_{ai} and ξ_{bi} have poles of order three at the critical layer. Thus the singularities in the primary flow give rise to more serious difficulties for the secondary flow. Clearly this singular behaviour can be corrected by using a viscous correction near $y = 0$.

Corresponding to the neutral oscillation $\hat{v}_{1i} = \operatorname{sech} y$, the equivalent two-dimensional component of this oscillation is given by $i\alpha\hat{u}_{1i} + \beta\hat{w}_{1i} = \operatorname{sech} y \tanh y$. However, the cross-wave component is $\hat{\phi}_i = \beta\hat{u}_{1i} + i\alpha\hat{w}_{1i} = +(i\beta/\alpha)\operatorname{sech}^3 y \coth y$, and has a pole at $y = 0$. The complete differential equation for $\hat{\phi}$ is obtained by taking a suitable combination from the set (2.15). It is

$$\frac{d^2\hat{\phi}}{dy^2} - [i\alpha R \tanh y + 1]\,\hat{\phi} = \beta R \operatorname{sech}^3 y. \tag{4.8}$$

Formally putting R infinite, we have the inviscid solution

$$\hat{\phi}_i = -\frac{\beta}{i\alpha}\operatorname{sech}^3 y \coth y. \tag{4.9}$$

It is well known that the critical layer has a thickness of order $(\alpha R)^{-\frac{1}{3}}$ and it is within this region that the solution $\hat{\phi}_i$ will be modified by a correction of the boundary-layer type.

A formal method of dealing with a homogeneous second-order differential equation, involving a large parameter αR, whose coefficient has a turning point, has been given by Langer (1949). The extension to the non-homogeneous problem is very simple. We sketch the method as it applies to the problem at hand.

First consider the general solution $\hat{\psi}$ of the homogeneous equation

$$\frac{d^2\hat{\psi}}{dy^2} - [i\alpha R \tanh y + 1]\,\hat{\psi} = 0. \tag{4.10}$$

Let
$$T(y) = \int_0^y \sqrt{(\tanh y)}\,dy = \tanh^{-1}\sqrt{(\tanh y)} - \tan^{-1}\sqrt{(\tanh y)}, \tag{4.11}$$

$$Z(y, \alpha R) = iY(y, \alpha R) = i(\alpha R)^{\frac{1}{3}}(3T/2)^{\frac{2}{3}}, \tag{4.12}$$

$$Q(y) = (3T/2)^{\frac{1}{6}}\left(\frac{dT}{dy}\right)^{-\frac{1}{2}}, \tag{4.13}$$

$$\hat{\psi}_1(y) = Q(y)\,h(Z), \tag{4.14}$$

where
$$\frac{d^2h}{dZ^2} + Zh = 0. \tag{4.15}$$

A straightforward substitution then shows that $\hat{\psi}_1$ satisfies the equation

$$\frac{d^2\hat{\psi}_1}{dy^2} - \left[i\alpha R \tanh y + \frac{1}{Q}\frac{d^2Q}{dy^2}\right]\hat{\psi}_1 = 0, \tag{4.16}$$

$Q^{-1}d^2Q/dy^2$ being regular and non-zero in a neighbourhood of $y = 0$. Equation (4.16) is considered as the approximating differential equation for (4.10) and correct to order $(\alpha R)^{-\frac{1}{3}}$ we may write $\hat{\psi} \equiv \hat{\psi}_1$.

The solutions of equation (4.15) are the modified Hankel functions of order one-third. That is, in the usual notation,

$$h_1(Z) = (\tfrac{2}{3}Z^{\frac{3}{2}})^{\frac{1}{3}} H_{\frac{1}{3}}^{(1)}(\tfrac{2}{3}Z^{\frac{3}{2}}), \tag{4.17}$$

$$h_2(Z) = (\tfrac{2}{3}Z^{\frac{3}{2}})^{\frac{1}{3}} H_{\frac{1}{3}}^{(2)}(\tfrac{2}{3}Z^{\frac{3}{2}}). \tag{4.18}$$

As $Q(y)$ is a known function of y, we can write the general solution of equation (4.10) as

$$\hat{\psi} = \{A_1 h_1(Z) + A_2 h_2(Z)\} Q(y), \tag{4.19}$$

where A_1 and A_2 are arbitrary constants.

Returning now to the non-homogeneous problem, the solution $\hat{\phi}(y)$ is required to approach the inviscid solution $\hat{\phi}_i(y)$ for $(\alpha R)^{\frac{1}{3}} y$ large. An application of the method of variation of constants readily shows that this solution is

$$\hat{\phi} = -\frac{\beta}{i\alpha} \operatorname{sech}^3 y \coth y \, ZL(Z), \tag{4.20}$$

where $L(Z)$ is a Lommel function, and is the solution of the differential equation $d^2L/dZ^2 + ZL = 1$, which behaves like Z^{-1} for $|Z|$ large. $L(Z)$ can easily be checked to be expressible in the form

$$L(Z) = \frac{h_2(Z)\int_{i\infty}^{Z} h_1(\zeta)\,d\xi - h_1(Z)\int_{-i\infty}^{Z} h_2(\zeta)\,d\zeta}{W\{h_1(Z), h_2(Z)\}}, \tag{4.21}$$

provided $-\frac{2}{3}\pi < \arg(Z) < \frac{2}{3}\pi$. In the above application $\arg(Z) = \pm\frac{1}{2}\pi$. $W\{h_1(Z), h_2(Z)\}$ denotes the Wronskian of the functions $h_1(Z)$ and $h_2(Z)$. In the usual normalization for the Hankel functions, we have

$$W\{h_1(Z), h_2(Z)\} = -\frac{4i}{\pi}\left(\frac{3}{2}\right)^{\frac{1}{3}}. \tag{4.22}$$

The above method of approximate solution can be systematically extended to higher approximations by using perturbation series in $(\alpha R)^{-\frac{1}{3}}$.

In dealing with the case of high Reynolds numbers we shall retain only the leading terms in αR for all quantities in the subsequent work. The viscous corrections, which apply for $Y = O(1)$, or $y = O(\alpha R)^{-\frac{1}{3}}$, remove the singularities obtained by the inviscid analysis.

The Lommel function plays an important role in the problem and it is useful to list a few of its properties. We write

$$L(Z) = L_r(Y) + iL_i(Y), \tag{4.23}$$

where

$$Y = (\alpha R)^{\frac{1}{3}} y. \tag{4.24}$$

From the differential equation we have

$$\frac{d^2L_r}{dY^2} + YL_i = -1, \tag{4.25}$$

$$\frac{d^2L_i}{dY^2} - YL_r = 0. \tag{4.26}$$

Also, using the result that $[h_1(Z^*)]^* = h_2(Z)$, it can be shown, from (4.21), that L_r is an even and L_i an odd function of Y. For Y large,

$$L_r = -\frac{2}{Y^4} + \frac{2240}{Y^{10}} - \ldots, \tag{4.27}$$

$$L_i = -\frac{1}{Y} + \frac{40}{Y^7} - \ldots. \tag{4.28}$$

The graphs of $L_r(Y)$ and $L_i(Y)$ are shown in figure 1.

Dropping the subscripts i for the corrected solution, a little manipulation yields

$$\left.\begin{aligned} \hat{u}_1 &= -\frac{\beta^2(\alpha R)^{\frac{1}{3}}}{\alpha} L_r(Y) - \frac{i}{\alpha}(\beta^2 \operatorname{cosech} y - \operatorname{sech} y \tanh y)\, Y L_i(Y), \\ \hat{v}_1 &= \operatorname{sech} y, \\ \hat{w}_1 &= -\beta \operatorname{cosech} y\, Y L_i(Y) + i\beta(\alpha R)^{\frac{1}{3}} L_r(Y), \end{aligned}\right\} \quad (4.29)$$

$$\left.\begin{aligned} \hat{U}_1 &= -i \operatorname{sech} y \tanh y, \\ \hat{V}_1 &= \operatorname{sech} y. \end{aligned}\right\} \quad (4.30)$$

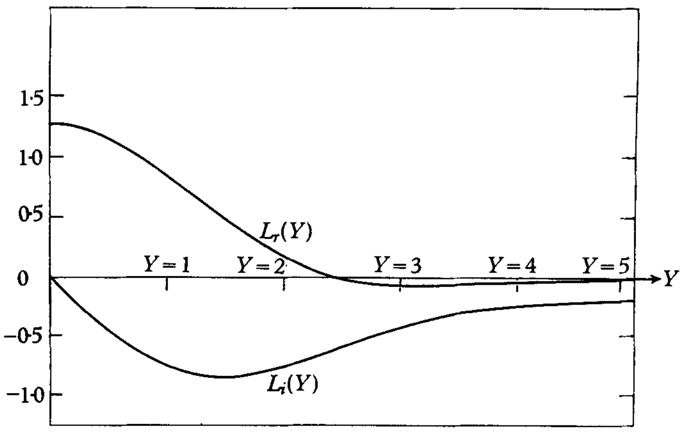

FIGURE 1. Graphs of $L_r(Y)$ and $L_i(Y)$.

The corresponding second-order quantities are

$$\left.\begin{aligned} \hat{\xi}_a &= -\frac{16\beta}{\sinh^3 2y}[(1-\beta^2)\cosh 2y - \beta^2]\, G(Y), \\ \hat{r}_a &= \frac{\beta^2(\alpha R)^{\frac{2}{3}}}{\alpha}\frac{dL_r}{dY}, \end{aligned}\right\} \quad (4.31)$$

where
$$G(Y) = \frac{Y^3}{2(1-2\beta^2)}\left[2\beta^2 \frac{d}{dY}(L_r^2 + L_i^2) - Y L_r\right];$$

$$\left.\begin{aligned} \hat{\xi}_b &= -\frac{2\beta}{\sinh^3 2y}[16\cosh 2y + \beta^2 \cosh 4y - \beta^2]\, H(Y), \\ \hat{r}_b &= \frac{2\beta^2(\alpha R)^{\frac{2}{3}}}{\alpha}\frac{dL_r}{dY}, \end{aligned}\right\} \quad (4.32)$$

where $H(Y) = -\frac{1}{2}Y^4 L_r$. Also,

$$\left.\begin{aligned} \hat{\xi}_c &= 0, \\ \hat{r}_c &= \frac{\beta^2(\alpha R)^{\frac{2}{3}}}{\alpha}\frac{dL_r}{dY}; \end{aligned}\right\} \quad (4.33)$$

$$\left.\begin{aligned} \hat{\xi}_d &= 0, \\ \hat{r}_d &= 0. \end{aligned}\right\} \quad (4.34)$$

For Y large

$$G(Y) = 1 - \frac{16(70-19\beta^2)}{(1-2\beta^2)\,Y^6} + \ldots, \tag{4.35}$$

$$H(Y) = 1 - \frac{1120}{Y^6} + \ldots. \tag{4.36}$$

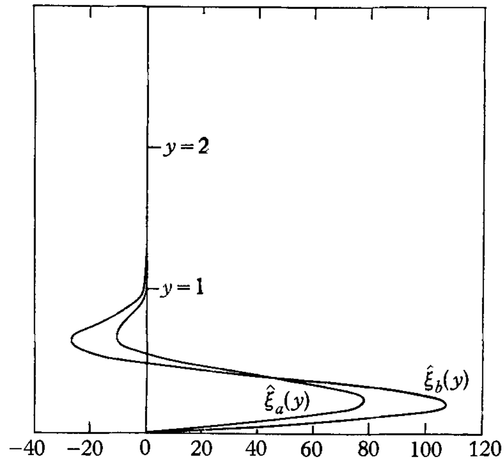

FIGURE 2. Mean secondary vorticity amplitude functions.

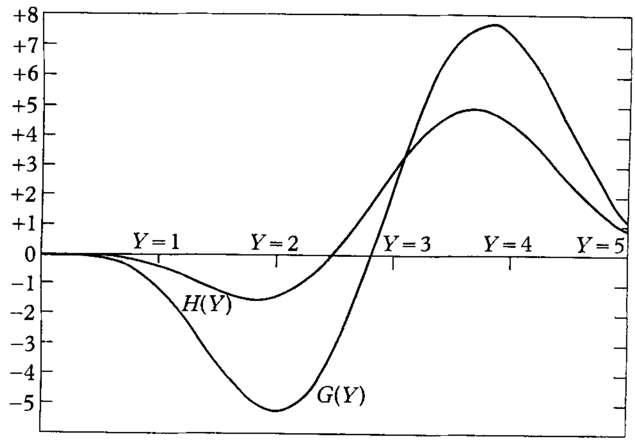

FIGURE 3. Correction functions $G(Y)$ and $H(Y)$.

The actual calculations have been performed for $\beta = \frac{1}{2}$, $\alpha R = 125$, which have been taken as typical of the behaviour of the functions. Both $\hat{\xi}_a$ and $\hat{\xi}_b$ are odd functions of y. Figure 2 shows these functions. Figure 3 shows the graphs of the correction functions $G(Y)$ and $H(Y)$.

We now apply these results and calculate the mean secondary motion.

5. The mean secondary motion

First we discuss the a flow contribution to the mean secondary motion. This has a spanwise period π/β, and is induced by the three-dimensional standing wave part of the primary oscillation. From (3.21) we have

$$\frac{d^2\hat{v}_a}{dy^2} - 4\beta^2\hat{v}_a = -2\beta\hat{\xi}_a, \tag{5.1}$$

$$\hat{w}_a = -\frac{1}{2\beta}\frac{d\hat{v}_a}{dy}, \tag{5.2}$$

where, from (4.31),

$$\hat{\xi}_a(y) = \xi_{ai}(y)\, G(Y). \tag{5.3}$$

We require the solution of (5.1), for $y > 0$, subject to the conditions $\hat{v}_a(0) = 0$, and $\hat{v}_a(y) \to 0$ as $y \to \infty$. For $y < 0$, $\hat{v}_a(y)$ and $\hat{w}_a(y)$ are defined as odd and even functions respectively. Thus

$$\hat{v}_a(-y) = -\hat{v}_a(y); \quad \hat{w}_a(-y) = \hat{w}_a(y). \tag{5.4}$$

It is to be noted in passing that the solution of (5.1) with $\hat{\xi}_a = 0$ gives two-dimensional potential type motions.

The solution of (5.1) satisfying the condition at infinity can be written down in explicit form by the method of variation of constants. We have

$$\hat{v}_a(y) = A\, e^{-2\beta y} - \int_\infty^y \hat{\xi}_a(t) \sinh 2\beta(y-t)\, dt, \tag{5.5}$$

$$\hat{w}_a(y) = A\, e^{-2\beta y} + \int_\infty^y \hat{\xi}_a(t) \cosh 2\beta(y-t)\, dt, \tag{5.6}$$

where both integrals are clearly convergent. For $\hat{v}_a(0) = 0$, A is determined from

$$A = \int_0^\infty \hat{\xi}_a(t) \sinh 2\beta t\, dt, \tag{5.7}$$

and so

$$w_a(0) = -\int_0^\infty \hat{\xi}_a(t)\, e^{-2\beta t} dt. \tag{5.8}$$

Using this value for A, we can rewrite $\hat{v}_a(y)$ and $\hat{w}_a(y)$ in an alternative form, convenient for small values of y, namely,

$$\hat{v}_a(y) = \left(\int_0^\infty \hat{\xi}_a(t)\, e^{-2\beta t} dt\right) \sinh 2\beta y - \int_0^y \hat{\xi}_a(t) \sinh 2\beta(y-t)\, dt, \tag{5.9}$$

$$\hat{w}_a(y) = -\left(\int_0^\infty \hat{\xi}_a(t)\, e^{-2\beta t} dt\right) \cosh 2\beta y + \int_0^y \hat{\xi}_a(t) \cosh 2\beta(y-t)\, dt. \tag{5.10}$$

Approximate values for the constants A and $\hat{w}_a(0)$ can be obtained in the following way. $\xi_{ai}(t)$ can be expanded as a Laurent series about $t = 0$ of the form

$$\xi_{ai}(t) = \frac{a_0}{t^3} + \frac{a_1}{t} + \ldots, \tag{5.11}$$

where, for the case at hand,

$$a_0 = -2\beta(1-2\beta^2). \tag{5.12}$$

Putting $T = (\alpha R)^{\frac{1}{3}} t$, and $Y = (\alpha R)^{\frac{1}{3}} y$, we have from equations (5.7), (5.8)

$$A = -4\beta^2(1-2\beta^2)(\alpha R)^{\frac{1}{3}} \int_0^\infty \frac{1}{T^2} G(T)\, dT + O(\alpha R)^{-\frac{1}{3}}, \tag{5.13}$$

and

$$\hat{w}_a(0) = 2\beta(1-2\beta^2)(\alpha R)^{\frac{2}{3}} \int_0^\infty \frac{1}{T^3} G(T)\, dT + O(\alpha R)^{\frac{1}{3}}. \tag{5.14}$$

On using the value for $G(T)$ from equation (4.31) we find, after some simplification,

$$A = 4(\alpha R)^{\frac{1}{3}} \beta^4 \int_0^\infty (L_r^2 + L_i^2)\, dT + O(\alpha R)^{-\frac{1}{3}}, \tag{5.15}$$

$$\hat{w}_a(0) = -(\alpha R)^{\frac{2}{3}} \beta \left[2\beta^2 L_r^2(0) - \frac{dL_i}{dY}(0)\right] + O(\alpha R)^{\frac{1}{3}}. \tag{5.16}$$

So for R large, $A > 0$, and $\hat{w}_a(0) < 0$.

Also from equation (5.9), using the same transformations, we have

$$\hat{v}_a(y) = \left\{\int_0^\infty \xi_a\left[\frac{T}{(\alpha R)^{\frac{1}{3}}}\right] \exp\left[-\frac{2\beta T}{(\alpha R)^{\frac{1}{3}}}\right] \frac{1}{(\alpha R)^{\frac{1}{3}}}\, dT\right\} \sinh\left[\frac{2\beta Y}{(\alpha R)^{\frac{1}{3}}}\right] - \int_0^Y \xi_a\left[\frac{T}{(\alpha R)^{\frac{1}{3}}}\right] \sinh\left[\frac{2\beta(Y-T)}{(\alpha R)^{\frac{1}{3}}}\right] \frac{1}{(\alpha R)^{\frac{1}{3}}}\, dT, \tag{5.17}$$

and so for Y large,

$$\hat{v}_a(y) \to -4\beta^2(1-2\beta^2)(\alpha R)^{\frac{1}{3}} \int_0^\infty \frac{1}{T^2} G(T)\, dT. \tag{5.18}$$

Thus $\hat{v}_a(y)$ tends to a constant value of order $(\alpha R)^{\frac{1}{3}}$ as we approach the outer edge of the critical layer. On comparing (5.13) and (5.18), it is seen that it is this term that induces the external potential motion $A\, e^{-2\beta y}$.

The preceding analysis shows very clearly the importance of the critical layer. Although viscous forces are negligible outside the small region $y = O(\alpha R)^{-\frac{1}{3}}$, we see that they are indeed responsible for inducing a potential component to the secondary motion, which is dominant far away from the critical layer. This two-dimensional secondary flow, induced by the purely three-dimensional part of the original oscillation, may be described as being due to a source distribution at the critical layer of strength $\sigma(z) = (A/\pi)\cos 2\beta z$ per unit length. This source distribution is of $O\{a^2\lambda^2(\alpha R)^{\frac{1}{3}}\}$, and determines the direction of the circulation for the a motion at infinity.

In figure 4 we show the graphs of $\hat{v}_a(y)$, $\hat{w}_a(y)$, $\hat{r}_a(y)$, and $\hat{s}_a(y)$.

By the same method we can discuss the b flow contribution to the mean secondary motion. This has a spanwise period $2\pi/\beta$ and is induced by the non-linear interaction of the two- and three-dimensional parts of the oscillation. From (3.22), we have

$$\frac{d^2\hat{v}_b}{dy^2} - \beta^2\hat{v}_b = -\beta\xi_b, \tag{5.19}$$

$$\hat{w}_b = -\frac{1}{\beta}\frac{d\hat{v}_b}{dy}, \tag{5.20}$$

$$\xi_b(y) = \xi_{bi}(y)\, H(Y). \tag{5.21}$$

These equations are of the same form as (5.1), (5.2), (5.3), with a, G, and 2β, replaced by b, H, and β, respectively.

Hence we find, for $y > 0$,

$$\hat{v}_b(y) = Be^{-\beta y} - \int_\infty^y \hat{\xi}_b(t)\sinh\beta(y-t)\,dt, \tag{5.22}$$

$$\hat{w}_b(y) = Be^{-\beta y} + \int_\infty^y \hat{\xi}_b(t)\cosh\beta(y-t)\,dt. \tag{5.23}$$

As before,

$$B = \int_0^\infty \hat{\xi}_b(t)\sinh\beta t\,dt, \tag{5.24}$$

$$\hat{w}_b(0) = -\int_0^\infty \hat{\xi}_b(t)\,e^{-\beta t}\,dt. \tag{5.25}$$

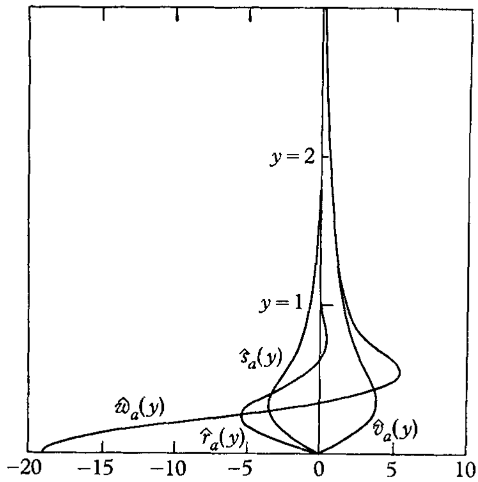

FIGURE 4. Amplitude functions for the a flow.

On using this value for B, the expressions for $\hat{v}_b(y)$ and $\hat{w}_b(y)$ can be rewritten in the alternative form

$$\hat{v}_b(y) = \left(\int_0^\infty \hat{\xi}_b(t)\,e^{-\beta t}dt\right)\sinh\beta y - \int_0^y \hat{\xi}_b(t)\sinh\beta(y-t)\,dt, \tag{5.26}$$

$$\hat{w}_b(y) = -\left(\int_0^\infty \hat{\xi}_b(t)\,e^{-\beta t}dt\right)\cosh\beta y + \int_0^y \hat{\xi}_b(t)\cosh\beta(y-t)\,dt. \tag{5.27}$$

Near $t = 0$,

$$\hat{\xi}_{bi}(t) = -\frac{4\beta}{t^3} + O\left(\frac{1}{t}\right), \tag{5.28}$$

and so

$$B = -4\beta^2(\alpha R)^{\frac{1}{3}}\int_0^\infty \frac{1}{T^2}H(T)\,dT + O(\alpha R)^{-\frac{1}{3}}, \tag{5.29}$$

$$\hat{w}_b(0) = 4\beta(\alpha R)^{\frac{2}{3}}\int_0^\infty \frac{1}{T^3}H(T)\,dT + O(\alpha R)^{\frac{1}{3}}. \tag{5.30}$$

On using the fact that $H(T) = -\frac{1}{2}T^4L_2(T)$, we find

$$B = O(\alpha R)^{-\frac{1}{3}}, \tag{5.31}$$

$$\hat{w}_b(0) = 2\beta(\alpha R)^{\frac{2}{3}}\frac{dL_i(0)}{dY} + O(\alpha R)^{\frac{1}{3}}. \tag{5.32}$$

Thus the interaction flow, or b flow, produces a potential component $O\{a^2\lambda\mu(\alpha R)^{-\frac{1}{3}}\}$, in contrast to the a flow. The same strong spanwise momentum exchange is present near the critical layer. Figure 5 shows the graphs of $\hat{v}_b(y)$, $\hat{w}_b(y)$, $\hat{r}_b(y)$ and $\hat{s}_b(y)$.

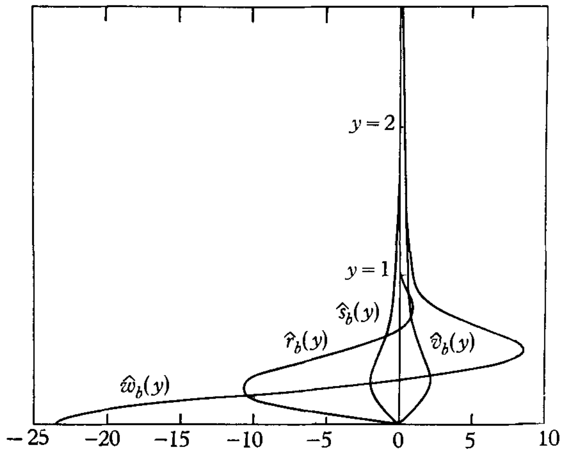

FIGURE 5. Amplitude functions for the b flow.

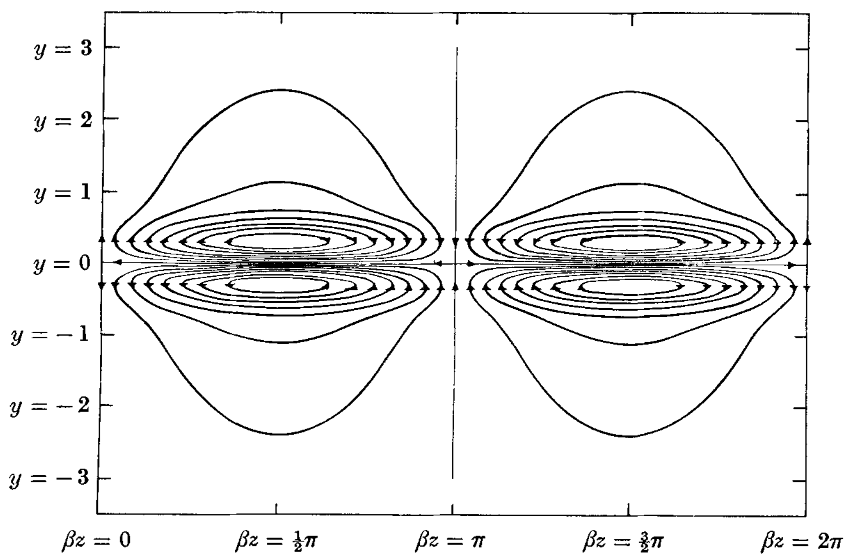

FIGURE 6. Streamline pattern for secondary motion, $\mu/\lambda \gg 1$.

The c flow part of the mean secondary flow is a purely two-dimensional distortion due, however, to the three-dimensionality of the original oscillation. The only non-zero second-order term is $\hat{r}_c(y) = \hat{r}_a(y)$, relating to the downstream velocity component. This tends to produce a defect in x-momentum close to the critical layer at all points across the profile, and a very slight excess at the outer edge of the critical layer. The effect is confined to the region $y = O(\alpha R)^{-\frac{1}{3}}$.

To the order of the approximation used in the present analysis, the d flow, due to the two-dimensional oscillation, does not affect the mean flow.

In order to interpret these results, it is desirable to plot the projections, on the (y, z)-plane, of the streamlines for the entire second-order mean motion. These will be obtained by solving the differential equation

$$\frac{dy}{v_0^{(2)}(y,z,t)} = \frac{dz}{w_0^{(2)}(y,z,t)}, \tag{5.33}$$

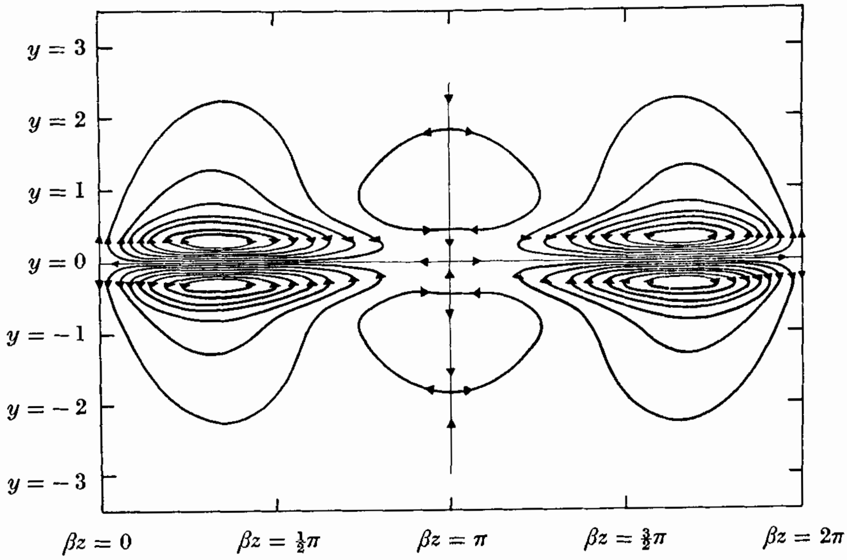

FIGURE 7. Streamline pattern for secondary motion, $\mu/\lambda = 2$.

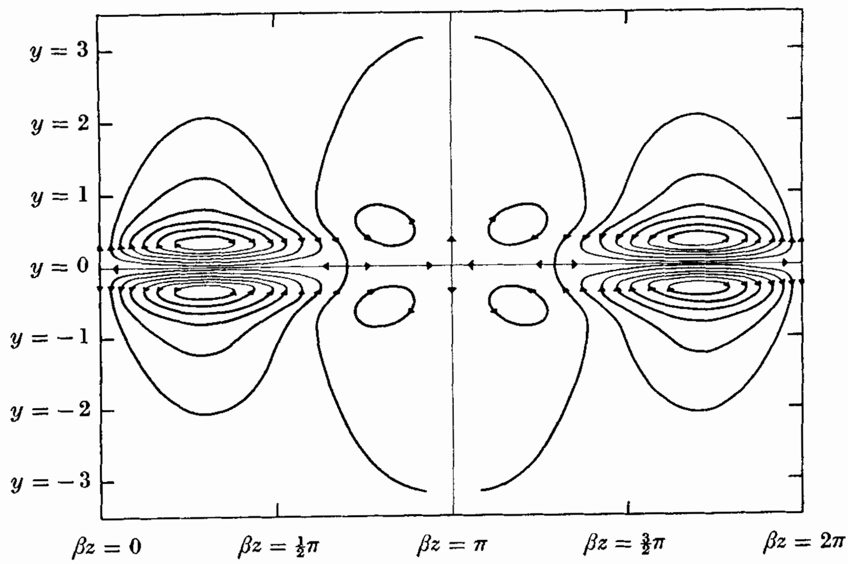

FIGURE 8. Streamline pattern for secondary motion, $\mu/\lambda = 1$.

that is, from

$$\frac{dy}{\lambda^2\hat{v}_a\cos 2\beta z+\lambda\mu\hat{v}_b\cos\beta z}=\frac{dz}{\lambda^2\hat{w}_a\sin 2\beta z+\lambda\mu\hat{w}_b\sin\beta z},\tag{5.34}$$

and so, on integration, the streamlines are given by

$$\frac{\lambda^2}{2\beta}\hat{v}_a\sin 2\beta z+\frac{\lambda\mu}{\beta}\hat{v}_b\sin\beta z=\text{const.}\tag{5.35}$$

Figures 6, 7, 8, 9, 10 show these streamline patterns for various ratios of μ/λ. These streamlines have been plotted using equal increments of the stream-function

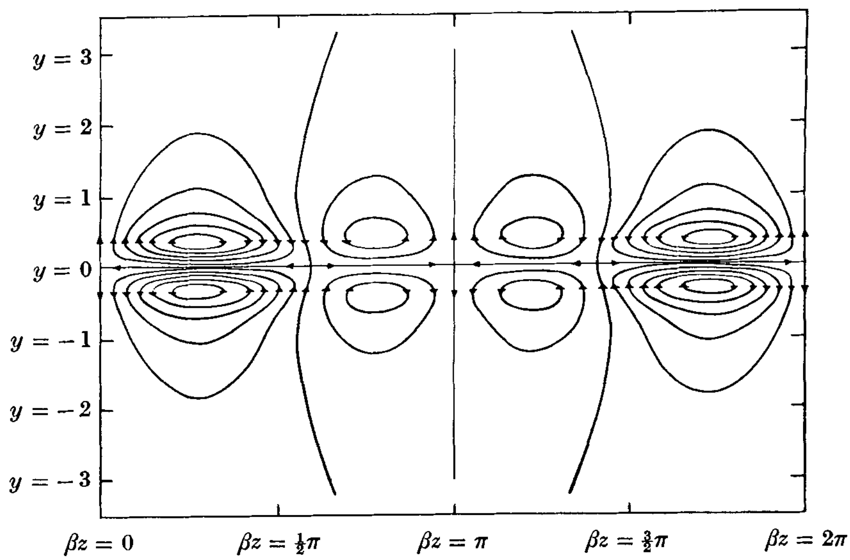

FIGURE 9. Streamline pattern for secondary motion, $\mu/\lambda = \frac{1}{2}$.

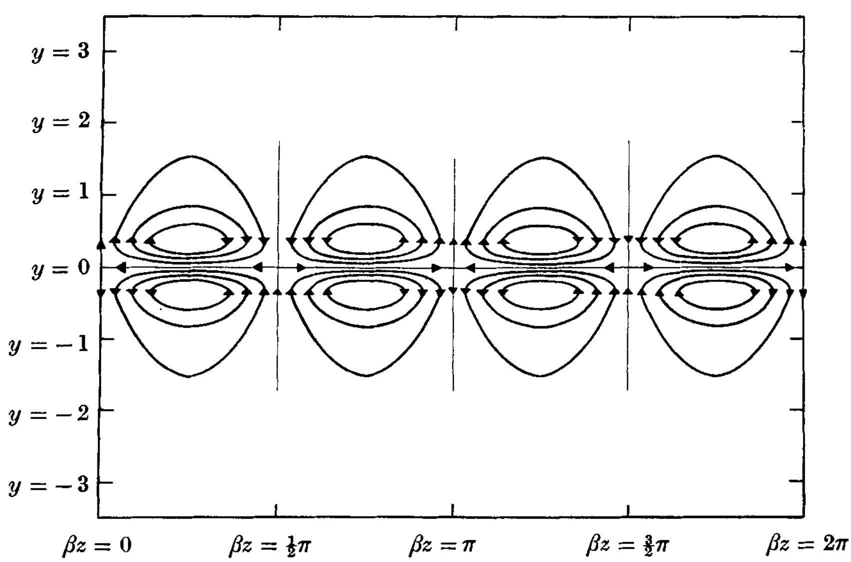

FIGURE 10. Streamline pattern for secondary motion, $\mu/\lambda = 0$.

from $y = 0$, and show clearly the steady longitudinal vortex structure of the secondary flow.

This completes the formal mathematical calculations for the mean flow. It remains to interpret these results and to discuss the physical mechanisms involved. To this purpose it is necessary to make a few simple observations on the primary oscillation.

6. The primary oscillation

It is to be anticipated that the effect of the secondary motion will be diminished or intensified by the downstream periodicity of the primary oscillation. The longitudinal vorticity associated with the three-dimensionality of the oscillation

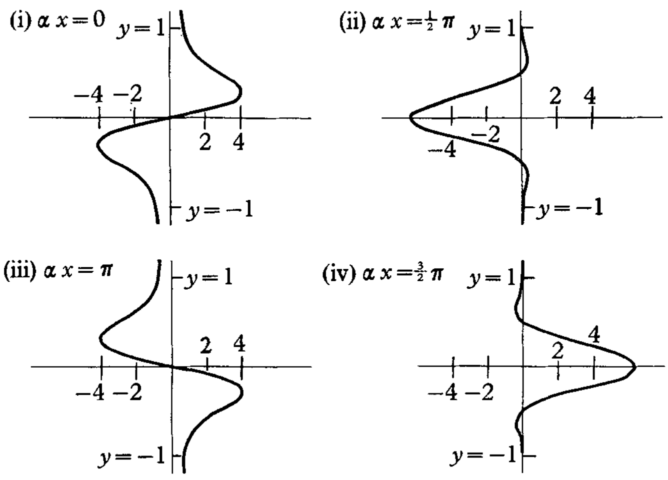

FIGURE 11. Amplitude of $w^{(1)}$ at spanwise position, $\beta z = \frac{1}{2}\pi$.

is of particular interest. Therefore we list the relevant fluctuations. They are, putting $c = 0$,

$$u^{(1)} = 2\lambda\left[-\frac{\beta^2(\alpha R)^{\frac{1}{3}}}{\alpha} L_r(Y)\cos\alpha x + \frac{1}{\alpha}(\beta^2 \operatorname{cosech} y - \operatorname{sech} y \tanh y)\, Y L_i(Y)\sin\alpha x\right]\cos\beta z + 2\mu[\operatorname{sech} y \tanh y \sin\alpha x], \tag{6.1}$$

$$v^{(1)} = 2\lambda[\operatorname{sech} y\cos\alpha x]\cos\beta z + 2\mu[\operatorname{sech} y\cos\alpha x], \tag{6.2}$$

$$w^{(1)} = 2\lambda[-\beta\operatorname{cosech} y\, YL_i(Y)\cos\alpha x - \beta(\alpha R)^{\frac{1}{3}}L_r(Y)\sin\alpha x]\sin\beta z, \tag{6.3}$$

$$\xi^{(1)} = 2\lambda\left[-\beta\operatorname{cosech}^2 y \operatorname{sech} y\, Y^2\frac{dL_i(Y)}{dY}\cos\alpha x - \beta(\alpha R)^{\frac{2}{3}}\frac{dL_r(Y)}{dY}\sin\alpha x\right]\sin\beta z. \tag{6.4}$$

It is seen, therefore, that $w^{(1)}$ and $\xi^{(1)}$ have spanwise maxima where βz is an odd multiple of $\frac{1}{2}\pi$. The distributions for $w^{(1)}$ and $\xi^{(1)}$, at the position $\beta z = \frac{1}{2}\pi$, are shown in figures 11 and 12, for different downstream positions. It will be noticed that when $\alpha x = \frac{1}{2}\pi$, the oscillation will lend support to the vortex distribution of the b flow. This corresponds to the convex part of the streamline. When $\alpha x = 3\pi/2$ the vorticity due to the oscillation tends to cancel that of the b flow. We shall return to this point in the next section.

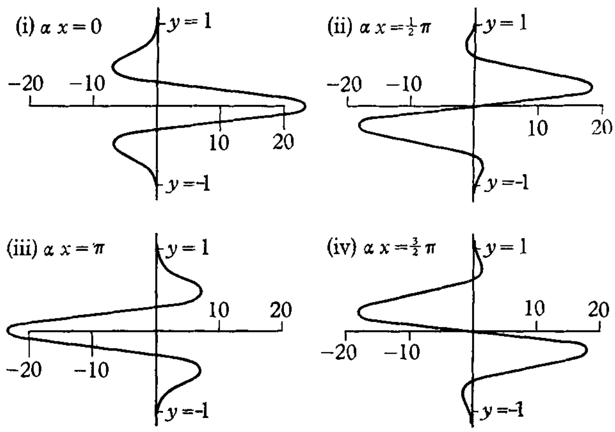

FIGURE 12. Amplitude of $\xi^{(1)}$ at spanwise position, $\beta z = \frac{1}{2}\pi$.

7. Physical interpretation of the results

In previous sections calculations have been made for the second-order velocities induced by oscillations of finite amplitude. It is clear that this mechanism in which the secondary vorticity produces a spanwise redistribution of momentum will apply to non-linear oscillations in any parallel flow, although the phase relationships may differ from case to case. Therefore, in this discussion, we shall restrict ourselves to an interpretation in the case of the profile $u_0(y) = \tanh y$. Also, in this report we shall neglect the effect of the second and higher harmonics.

Initially one must expect that the oscillation will be mainly two-dimensional, that is $\mu/\lambda \gg 1$, and so the steady secondary flow will be essentially the b flow. This b flow induces a steady cellular vortex structure on the motion as indicated in figure 6. Momentum is extracted from the viscous region at spanwise positions $\beta z = 0, \pm 2\pi, \pm 4\pi, \ldots$, where the amplitude of the $u^{(1)}$ oscillation is a maximum, and fed back into this region at intermediate spanwise positions $\beta z = \pm\pi, \pm 3\pi, \ldots$. This large-scale exchange process produces an alternate defect and excess of x-momentum at these points respectively. In turn it is responsible for a spanwise alternate thinning and bulging of the original velocity profile. From figure 5, it is seen that this effect should be most pronounced at the outer edges of the critical layer, the profile gradually becoming more and more warped.

This picture is somewhat modified by the primary oscillation itself. The oscillation is periodic in x and has the same spanwise period as that of the b flow. The vorticity distribution of the oscillation is an even or odd function of y according to whether αx is an even or odd multiple of $\frac{1}{2}\pi$. Clearly from figures 11 and 12 this vortex system will tend to reinforce and cancel that of the secondary b flow once each downstream wavelength. This vortex intensification will take place when $\alpha x = \frac{1}{2}\pi$, $\frac{5}{2}\pi$, $\frac{9}{2}\pi, \ldots$, corresponding to longitudinal vortices having their centres at spanwise positions $\beta z = \frac{1}{2}\pi$, $\frac{3}{2}\pi, \ldots$. At such points the streamline is convex, the maximum cancellation occurring where the streamline is concave, contrary to the Görtler & Witting model (1957). This does not mean that the Görtler & Witting mechanism is not present, but it does show that there are other effects beyond those indicated by their theory. Existing experimental evidence lends support to the idea of vortex intensification at the convex part of the streamline, as predicted by the present theory.

If the three-dimensionality of the oscillation increases, then the a flow contribution to the secondary motion will become more prominent. The extreme case for an entirely standing wave oscillation is shown in figure 10. This a flow also induces a cellular vortex structure on the motion, the spanwise period being π/β. The vorticity is again an odd function of y. However, this motion is unlikely to be observed without a considerable b flow contribution being superposed.

We now trace growth of the secondary motion as the three-dimensionality of the oscillation becomes stronger. The secondary flow goes through the stages sketched in figures 6 to 10. The original vortices move closer towards the velocity defect regions, and much weaker vortices begin to appear outside the critical layer near the excess positions. The centres of these latter vortices gradually move down closer to the critical layer as the motion becomes more three-dimensional. It is to be recalled that the a flow has a dominant potential component $O\{a^2\lambda^2(\alpha R)^{\frac{1}{3}}\}$. For the stronger vortices the same periodic intensification will be present at the downstream stations $\alpha x = \frac{1}{2}\pi$, $\frac{5}{2}\pi, \ldots$; but in the case of the weaker newly formed vortices, which may be difficult to observe, the reinforcement will be less marked. However, any such detectable intensification should now take place at $\alpha x = \frac{3}{2}\pi, \frac{7}{2}\pi, \ldots$.

This longitudinal vortex structure and the associated mechanism of momentum transfer produces spanwise alternate excesses and defects in the x-momentum. The directions of the a and b vortices is such as to accentuate the spanwise points of velocity defect, which should therefore be quite sharply defined at $\beta z = 0$, $\pm 2\pi$, $\pm 4\pi, \ldots$. The c flow adds a two-dimensional defect to the profile at all spanwise positions. Although the thinning of the profile should be quite pronounced, the bulging at the intermediate positions $\beta z = \pm\pi$, $\pm 3\pi, \ldots$ will be more evenly distributed about these points. In fact, as the wave becomes more warped, these regions of excess may show a small defect region at their centres. These conclusions can readily be seen by referring to the streamline patterns.

8. Concluding remarks

In this paper we have used formal mathematical methods to investigate finite-amplitude oscillations during the breakdown of laminar flow. It is to be emphasized that while we do not discount ordinary two-dimensional distortions, it is felt that in many situations the formation of secondary vortices and the associated crumbling of the profile will be more strongly evident. Certainly this mechanism must be present to some degree during transition. In recent experimental literature there is repeated reference to the formation of this vortex structure.

Clearly the theory presented here could be improved in accuracy by a slightly more sophisticated approach. For instance, no account has been taken of the small differences in the values of α, αc_r, and αc_i for the two- and three-dimensional oscillations. The difference in amplification rates could have been included very easily and is, in any case, compensated for by the disposable ratio μ/λ. The differences in α and αc_r have been neglected on the grounds that their effects over the few wavelengths of practical interest would be small. Indeed, one might anticipate that in the non-linear range a synchronization phenomenon would occur, similar to that encountered in the study of periodic solutions of ordinary differential equations. Another point to be mentioned is that the principal three-dimensional mean flow distortion (the b flow) is linear in the three-dimensional component of the primary oscillation, and so for the purposes of calculation any value of β will yield a typical secondary-flow structure.

Doubtless the most important practical situation is that of boundary-layer transition. Here the Reynolds number depends on the downstream position (as opposed to the case of channel flow), and experimentally the waves develop as a space amplification rather than as a time amplification. Thus a more realistic approach to this problem would be to follow a wave of given frequency (αc real), and to examine its amplification with x (α complex). This would require some modifications of the present theory. In the non-linear problem one would then have a second-order mean flow growing in x rather than in t.

During completion of this work Dr G. B. Schubauer and Mr P. S. Klebanoff, in a private communication, have very kindly forwarded the results of some recent unpublished experimental work on a Blasius profile, performed at the National Bureau of Standards. Despite the vast difference between the case of shear-flow and boundary-layer instability, there is a very distinct agreement with the present theory. A theoretical study of the non-linear effects during boundary-layer instability is planned in the near future. This will enable a detailed comparison of theory and experiment to be made.

The author wishes to express his gratitude to Professor C. C. Lin of the Massachusetts Institute of Technology for his help and encouragement during this investigation. The general results in the first few sections of this paper were obtained jointly with Professor Lin, and have been summarized elsewhere (Benney & Lin 1960). My thanks also to Dr G. B. Schubauer and Mr P. S. Klebanoff of the National Bureau of Standards for several pleasant and informative discussions, and to Dr J. T. Stuart and Mr J. Watson of the National Physical Laboratory for reading the manuscript. This work was supported in part by the Office of Naval Research.

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